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Unified Field Theory with Homotopic Charge

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Previously proposed field equations for the field ϕ which maps points in space–time to points on the two-sphere are derived from a suitable Lagrangian. The original conjecture that this theory may be the nonlinear theory of electrodynamics which has charge quantization as a topological property is supported by this result. Problems with this interpretation are indicated.

1. INTRODUCTION

In a previous paper (Pisello, 1977) we proposed a theory of a field which maps space–time into the two-sphere, subject to the boundary condition $\phi = \phi_0$ at spatial infinity, where ϕ_0 is a fixed point on the sphere, e.g., the north pole. This field admits homotopically invariant structures, first discovered by Hopf (1931), which we call "kinks," a term introduced by Finkelstein and Rubenstein (1968). Here we describe the kink. The antisymmetric tensor field F_{ij}, derivable from a vector potential A_i, is constructed from ϕ by using more compact notation. The class of gauge-dependent kink-current densities is then constructed from A_i and F_{ij} by applying the theorem of Whitehead (1947). Equations of motion are derived from the Lagrangian $L = \frac{1}{4}F^2$ and it is proved that these equations imply the current $J^i = \partial_j F^{ji}$ is a kink current. In Section 7 we point out some of the implications of the speculation made in the earlier paper that this theory may be the nonlinear modification of Maxwell–Lorentz electrodynamics, which Einstein suggested might lead to a quantum theory of radiation.

2. THE KINK

The two-sphere is mapped into a flat disk so that the south pole is identified with the center of the disk and the north pole is mapped into the

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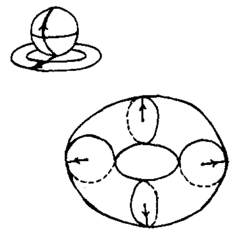

Fig. 1. The solid torus in physical space is mapped onto the surface of an abstract two-sphere. Each cross section of the torus is a disk which covers the sphere once.

rim. (See Figure 1.) The disk is then moved around so that the surface generated by the motion of the rim is a torus. As the disk moves once around the torus, it turns through an angle of 2π rad about the normal through the center of the disk in the direction of motion. The disk leaves behind the image of the sphere which it carries and thus defines the mapping inside the torus. All points on and outside the torus are mapped into the north pole. Such a mapping represents a single kink in the field ϕ. If the disk is rotated through an angle of -2π, the mirror image or antikink results. A kink and antikink have opposite homotopic charge; they annihilate each other.

3. THE TENSOR FIELD

Let a and b be coordinates on the sphere and $x = (x^0, x^1, x^2, x^3)$ be coordinates in flat space–time. The field $\phi(x)$ can be represented locally by a pair of continuous real-valued functions $a(x)$, $b(x)$. Let $M(a, b)$ define an area density on the sphere, so that the total area is unity. Introduce the symbols a_i, b_i to represent the derivatives of $a(x)$ and $b(x)$ with respect to the coordinates x^i. The tensor field F_{ij}, defined by

$$F_{ij} = M(a, b)(a_i b_j - a_j b_i) \tag{3.1}$$

is independent of the choice of coordinates on the sphere and satisfies the relations

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0 \tag{3.2}$$

familiar from the theory of the electromagnetic field. Therefore F_{ij} can be derived from a vector potential

$$\partial_i A_j - \partial_j A_i = F_{ij} \tag{3.3}$$

where A_i is determined up to the gradient of an arbitrary scalar gauge function. We call (a, b) flux coordinates because the "magnetic" field lies in the surfaces of constant a or b.

4. THE KINK CURRENT

Associated with each such A_i is a conserved pseudovector current defined by

$$K^i(A) = \tfrac{1}{2}\epsilon^{ijkl}A_jF_{kl} \tag{4.1}$$

where ϵ^{ijkl} is antisymmetric in all indices and $\epsilon^{0123} = 1$. According to Whitehead's theorem (1947), the integral of the time component of such a current over a region of space counts the number of kinks minus antikinks contained in that region. Hence we call $K^i(A)$ the kink-current density associated with the vector potential A_i.

5. THE EQUATIONS OF MOTION

Equations of motion for the field ϕ are derived from the Lagrangian density

$$L = \tfrac{1}{4}F_{ij}F^{ij} \tag{5.1}$$

by variation of $a(x)$ and $b(x)$. (The raising and lowering of indices is accomplished by contraction with the metric tensor g_{ij}, where $g_{ij} = 0$ for $i \neq j$, $-g_{00}\,g_{11} = g_{22} = g_{33} = -1$.) If we chose a and b so that $M(a, b) = 1$, then

$$\frac{\delta L}{\delta a} = \frac{\partial L}{\partial a} - \partial_i \frac{\partial L}{\partial a_i} = -\partial_i\left(\tfrac{1}{2}F^{kj}\frac{\partial}{\partial a_i}F_{kj}\right) = -\partial_i[\tfrac{1}{2}F^{kj}(b_j\delta_{ik} - b_k\delta_{ij})]$$

$$= -\partial_i(F^{ij}b_j) = -\partial_i F^{ij}b_j = -J^jb_j = 0$$

and similarly for $\delta L/\delta b = 0$. The resulting equations,

$$J^ia_i = 0 \tag{5.2a}$$

$$J^ib_i = 0 \tag{5.2b}$$

where $J^j = \partial_i F^{ij}$, are quasilinear second-order partial-differential equations for the functions $a(x)$ and $b(x)$. If $a_i(x)$ and $b_i(x)$ are specified on a timelike hypersurface S_k such that

$$\epsilon^{ijkl}a_ib_jS_k \neq 0 \tag{5.3}$$

then equations (5.2) are independent linear equations for the second time derivatives of a and b. This condition insures the integrability of (5.2), and F_{ij} may be regarded as determined over all space–time. In analogy to electromagnetism, J^i is called the current for F_{ij}.

6. THE CURRENT J AS A KINK CURRENT

Next we prove that J^i is a kink-current density. It suffices to show that there exists A_i^*, a solution of (3.3), which satisfies

$$J^i = eK^i(A^*) \tag{6.1}$$

where $e = \pm 1$, depending on the handedness of the coordinate system. Let $A_i^* = A_i + \partial_i Q$, where A_i is some known solution of (3.3), and the gauge function $Q(x)$ is the solution of the differential equation

$$J^i A_i + J^i \partial_i Q = J^i A_i^* = 0 \tag{6.2}$$

subject to the initial condition

$$J^i S_i = eK^i(A^*) S_i \tag{6.3}$$

on the timelike hypersurface S_i. If $K^i(A^*)$ does not vanish identically, then a_i, b_i, A_i^*, and $K^i(A^*)$ are linearly independent at each point in space–time. Equations (5.2) and (6.2) therefore imply

$$J^i = q(x) K^i(A^*) \tag{6.4}$$

where $q(x)$ is a pseudoscalar function. Since both J^i and K^i are conserved, $q(x)$ must satisfy the differential equation

$$K^i \partial_i q = 0 \tag{6.5}$$

while (6.3) specifies $q(x) = e$ on S_i. Integration of (6.5) yields $q(x) = e$ everywhere in space–time. Thus the desired result is obtained. Notice that J^i is a kink-current density which is a local function of the field ϕ.

7. SUMMARY AND DISCUSSION

We have presented a Lagrangian theory of the field ϕ which maps elements of area in space–time into elements of area on the two-sphere, $\phi\colon dx^i\, dx^j \to F_{ij}\, dx^i\, dx^j$. We have shown that there exists a vector potential A_i^*, such that

$$F_{ij} = \partial_i A_j^* - \partial_j A_i^* \qquad \text{and} \qquad \partial_i F^{ij} = \tfrac{1}{2} e \epsilon^{jklm} A_k^* F_{lm}$$

Thus F_{ij} obeys Maxwell's equations with electric charge quantized by virtue of its being identical to the homotopic charge of the theory. The interpretation of F_{ij} as the electromagnetic field presents difficult problems connected with the physical interpretation of the nonlinear field. However, the implications of such an interpretation are far reaching.

Equation (5.3) implies that the magnetic field may not vanish on S_i. Therefore a singly charged stationary solution will have an intrinsic magnetic moment. However, the Lagrangian is scale invariant, so that stationary solutions have no definite size. This indicates the need to complicate the theory by introducing a wider symmetry group. This question will be treated in a future publication.

The interpretation of F_{ij} as the electromagnetic field raises another immediate problem. The identity

$$F_{ij} F_{kl} \epsilon^{ijkl} = 0$$

implies that the "electric" and "magnetic" parts of F_{ij} are everywhere and always mutually perpendicular. However, the usual ideas about the electromagnetic field are derived from the operationally defined linear field and may not be applied to the nonlinear F_{ij}, since there is no test charge distinct from the field itself in the unified theory. Full understanding of the connection between the nonlinear electromagnetic field and the experimental facts of electrodynamics must await the development of a theory of measurement of the unified field. Such a theory must be capable of representing both the measuring apparatus and the measured system in terms of the same fundamental field.

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